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The Power Weighted Gompertz Model

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Abstract

In this paper, we introduced a new extension model of distribution. This model is the Power weighted Gompertz distribution (PWG), it is generated by the power transformation method. We obtained some statistical properties of the new model, including moments, moment generating function, some types of entropies, rsidual life and reversed residual life functions, and Bonferroni and Lorenz curves. Estimation of the parameters of extended distribution is obtained by the method of maximum likelihood. To check the usefulness of new model, we applied two real data set and used some goodness of fit statistics. We illustrated the versatility of proposed model to fit and model data and confirmed that this model provide a better fit than some other very well-known distributions.

Keywords: ***Power transformation; Gompertz distribution; moments; entropies; Maximum likelihood estimation.***

1. Introduction

The Gompertz distribution plays an important role in modeling reliability, survival times, human mortality and actuarial data that have hazard rate with exponential increase. Therefore, it has received considerable attention from demographers and description the distribution of adult life spans by actuaries (see Willemse and Koppelaar) [14]. Recently, Bakouch and Abd El-Bar [1] introduced a new version of the Gompertz distribution. This new distribution is known as weighted Gompertz (WGo) distribution which represents a mixture of classical Gompertz and second upper record value of Gompertz densities. The two parameters weighted Gompertz distribution has the cdf is given by

$$G(x) = 1 - \left(1 + \frac{\sigma\left(e^{\lambda x}-1\right)}{1+\lambda\sigma}\right) e^{-\sigma\left(e^{\lambda x}-1\right)}, x > 0; \lambda > 0, \sigma > 0. \qquad (1)$$

The corresponding pdf is defined as follows

$$g(x) = \frac{\lambda\sigma^2}{1+\lambda\sigma}\left(\lambda + e^{\lambda x} - 1\right) e^{\lambda x - \sigma\left(e^{\lambda x}-1\right)}. \qquad (2)$$

In this paper, we present a new distribution having three parameter which is based on the WGo defined by Eq. (1), so-called power weighted Gompertz (PWG) distribution. Then we present various properties of the PWG distribution such as moments, moment generating function, three popular entropies, some measures of residual lifetime and reversed residual lifetime, estimation of the distribution parameters with the observed information matrix. Also, the applicability of the PWG distribution is shown by considering two real data sets, and related measures are obtained for both the data sets under the PWG distribution.

1.1. The Proposed Model

In this subsection, we defined the power weighted Gompertz model. A random variable X is said to have a power distribution if its cdf satisfies the following relationship

$$F(x) = G(x^{\alpha}), \alpha > 0. \qquad (3)$$

where $G(x)$ is the cdf of the baseline distribution. Now, we introduce the PWG model by taking $G(x)$ in Eq. (3) to be the cdf (1) of the WGo distribution. Therefore, the cdf of the PWG distribution is

$$F(x) = 1 - \left(1 + \frac{\sigma\left(e^{\lambda x^{\alpha}}-1\right)}{1+\lambda\sigma}\right) e^{-\sigma\left(e^{\lambda x^{\alpha}}-1\right)}, x > 0; \lambda > 0, \sigma > 0, \alpha > 0. \qquad (4)$$

The pdf of the PWG model is

$$f(x) = \frac{\lambda\alpha\sigma^2}{1+\lambda\sigma} x^{\alpha-1}\left(\lambda + e^{\lambda x^\alpha} - 1\right)e^{\lambda x^\alpha - \sigma\left(e^{\lambda x^\alpha}-1\right)}. \quad (5)$$

As a result of (4) and (5), the hazard rate function of PWG can be defined as

$$h(x) = \lambda\alpha\sigma^2 x^{\alpha-1} \frac{\left(\lambda+e^{\lambda x^\alpha}-1\right)e^{\lambda x^\alpha}}{1+\left(\lambda+e^{\lambda x^\alpha}-1\right)\sigma}. \quad (6)$$

1.2. Motivations of the Study

The motivations of this study are to present PWG due to the following:

- It is observed that the density function of the new model provides a wide range of shapes based on its additional shape parameter, for example a decreasing density of WGo will become monotonically decreasing, decreasing, symmetric, right-skewed and left-skewed (see Figure 1).
- The PWG distribution have decreasing, increasing and bathtub shaped hazard function based on its additional parameter and can be used to provide a good fit for the real data than well-known distributions (see Figure 1).
- The density of PWG can be obtained as a mixture of two positive ones, namely

$$f(x;\alpha,\lambda,\sigma) = \frac{\lambda\sigma}{1+\lambda\sigma} g_1(x;\alpha,\lambda,\sigma) + \frac{1}{1+\lambda\sigma} g_2(x;\alpha,\lambda,\sigma),$$

where $g_1(x;\alpha,\lambda,\sigma)$ and $g_2(x;\alpha,\lambda,\sigma)$ are the density functions of power Gompertz

and power 2nd upper record value of the Gompertz distribution defined as

$$g_1(x;\alpha,\lambda,\sigma) = \alpha\lambda\sigma x^{\alpha-1} e^{\lambda x^\alpha - \sigma\left(e^{\lambda x^\alpha}-1\right)},$$

and

$$g_2(x;\alpha,\lambda,\sigma) = \alpha\lambda\sigma^2 x^{\alpha-1}\left(e^{\lambda x^\alpha} - 1\right)e^{\lambda x^\alpha - \sigma\left(e^{\lambda x^\alpha}-1\right)},$$

respectively.

- Additionally, the new model contains some distributions as special cases, these sub-models being discussed as:

  - **Three-parameter power Lindley distribution (new).** Under Xhaving the PWG distribution and the transformation $Y^\beta = e^{\lambda X^\alpha} - 1$, the distribution of Y follows the three parameters power Lindley distribution with pdf

$$f(y) = \frac{\beta\sigma^2}{1+\lambda\sigma} y^{\beta-1}\left(\lambda + y^\beta\right)e^{-\sigma y^\beta}.$$

  - **Power Lindley distribution (see Ghitany et al. [5]).**

In fact, the particular case of (5) for $\lambda = 1$ and the transformation $Y^\beta = e^{\lambda X^\alpha} - 1$, is the power Lindley distribution with pdf

$$f(y) = \frac{\beta\sigma^2}{1+\sigma} y^{\beta-1}\left(1 + y^\beta\right)e^{-\sigma y^\beta}.$$

  - **Two parameter Lindley distribution (see Shanker et al. [12]).**

Let Xbe a continuously distributed random variable with density (5). Then the random variable $Y = e^{\lambda X^\alpha} - 1$has a two-parameter Lindley distribution with pdf

$$f(y) = \frac{\sigma^2}{1+\lambda\sigma}(\lambda + y)e^{-\sigma y}.$$

Further, in the above equation, when $\lambda = 1$, we have the Lindley distribution.

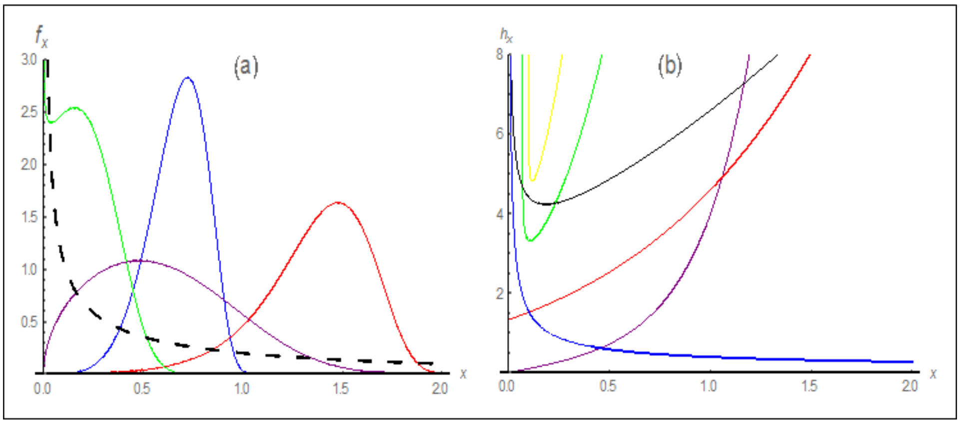

Figure 1: Plots of the PWG density and hazard functions.

(a) $\alpha = 0.3, \lambda = 1, \sigma = 1$ (dashes-black), $\alpha = 0.9, \lambda = 4, \sigma = 0.6$ (green), $\alpha = 1.5, \lambda = 0.5, \sigma = 4$ (purple), $\alpha = 4, \lambda = 0.1, \sigma = 3$ (red), $\alpha = 5, \lambda = 1, \sigma = 5$ (blue).
(b) $\alpha = 0.2, \lambda = 1, \sigma = 1$ (blue), $\alpha = 0.3, \lambda = 1, \sigma = 1$ (black), $\alpha = 0.8, \lambda = 3, \sigma = 0.8$ (green), $\alpha = 0.9, \lambda = 4, \sigma = 1$ (yellow), $\alpha = 1, \lambda = 1, \sigma = 2$ (red), $\alpha = 2, \lambda = 1, \sigma = 1$ (purple).

The rest of this paper is organized as follows. In Section 2, we introduce some important statistical and reliability measures for PWG model, including the moments, moment generating function, three types of entropies, some measures of residual life and reversed residual life functions such as density, survival and hazard rate functions with mean and variance and Bonferroni and Lorenz curves. In Section 3, estimation of the parameters of PWG model and the observed information matrix are verified. In Section 4, two real data sets are used to assess the performance of PWG model among some classical and recent distributions based on some evaluation goodness-of-fit statistics. Finally, some concluding remarks are made in Section 5.

2. Statistical and Reliability Properties

In this section, we introduced some important statistical and reliability measures for PWG model, including the moments, moment generating function, three types of entropies, Residual life and reversed residual life functions and Bonferroni and Lorenz curves.

2.1. Moments

The r^{th}moment of the PWG model is given by

$$\mu_r^/ = E(X^r) = \int_0^\infty x^r f_{PWG}(x)dx, \text{for } r = 1,2,..$$

$$= \sum_{k=0}^{\infty} \omega_{k,r} \left[\frac{\lambda-1}{(k+1)^{\frac{r}{\alpha}+1}} + \frac{1}{(k+2)^{\frac{r}{\alpha}+1}}\right], r = 1,2, \dots, \quad (7)$$

where $\omega_{k,r} = \frac{(-1)^{\frac{r}{\alpha}+k+1}\sigma^{k+2}e^{\sigma}}{(1+\lambda\sigma)\lambda^{\frac{r}{\alpha}}\Gamma(k+1)}\Gamma\left(\frac{r}{\alpha}+1\right)$.

In particular, first and second moment for PWG are given, respectively by

$$\mu_1^/ = \mu = \sum_{k=0}^{\infty} \frac{(-1)^{\frac{1}{\alpha}+k+1}\sigma^{k+2}e^{\sigma}}{(1+\lambda\sigma)\lambda^{\frac{1}{\alpha}}\Gamma(k+1)}\Gamma\left(\frac{1}{\alpha}+1\right)\left[\frac{\lambda-1}{(k+1)^{\frac{1}{\alpha}+1}} + \frac{1}{(k+2)^{\frac{1}{\alpha}+1}}\right], \quad (8)$$

$$\mu_2^/ = \sum_{k=0}^{\infty} \frac{(-1)^{\frac{2}{\alpha}+k+1}\sigma^{k+2}e^{\sigma}}{(1+\lambda\sigma)\lambda^{\frac{2}{\alpha}}\Gamma(k+1)}\Gamma\left(\frac{2}{\alpha}+1\right)\left[\frac{\lambda-1}{(k+1)^{\frac{2}{\alpha}+1}} + \frac{1}{(k+2)^{\frac{2}{\alpha}+1}}\right], \quad (9)$$

and then

$$Var(x) = \sigma^2 = \mu_2^/ - \mu^2. \quad (10)$$

2.2. Moment generating function

The moment generating function (m.g.f) of the PWG model is defined by

$$M_x(t) = \frac{\lambda\alpha\sigma^2 e^{\sigma}}{1+\lambda\sigma}\sum_{j=0}^{\infty}\frac{t^j}{j!}\frac{(-1)^{j/\alpha+1}}{\alpha\lambda^{j/\alpha+1}}\Gamma\left(\frac{j}{\alpha}+1\right)\sum_{k=0}^{\infty}\frac{(-1)^k\sigma^k}{\Gamma(k+1)}\left[\frac{(\lambda-1)}{(k+1)^{j/\alpha+1}} + \frac{1}{(k+2)^{j/\alpha+1}}\right]. \quad (11)$$

The mean and the variance of PWG model are presented in Table 1 for various values of α, σ and λ. It is observed that, both of them decreases as the values of parameters increase.

Table 1: Mean and variance for different values of α,σ and λ.

parameters	$\lambda = 0.8$ $\sigma = 0.1$	
α	Mean	Variance
1	3.44253	0.910962
1.5	2.25674	0.200146
2	1.83341	0.081136
2.5	1.62054	0.043116
parameters	$\lambda = 0.9$ $\alpha = 2$	
σ	Mean	Variance
0.6	1.08693	0.101591
0.9	0.935192	0.096780
1.2	0.831625	0.0898217
1.8	0.695632	0.076326
parameters	$\boldsymbol{\alpha = 1.5}$ $\boldsymbol{\sigma = 0.2}$	
λ	Mean	Variance
1.2	1.40777	0.135793
1.8	1.05455	0.085566
2.2	0.912566	0.0680568
2.8	0.765973	0.0515705

2.3. Entropies

Entropy is a measure of randomness of systems which is widely used in areas like physics, molecular imaging of tumors and sparse kernel density estimation. Three popular entropy measures are Shannon entropy , Rényi entropy and Mathai-Haubold entropy defined by:

$\eta_x = E(-\log f(x))$,

$I_R(\gamma) = \frac{1}{1-\gamma} \log\left(\int_{\mathbb{R}} f^{\gamma}(x)dx\right)$, where $\gamma > 0$ and $\gamma \neq 1$

and

$J_{MH}(\delta) = \frac{1}{\delta-1}\int_{\mathbb{R}}(f(x))^{2-\delta}dx - 1$, where $\delta < 2$ and $\delta \neq 1$

respectively.

2.3.1. Shannon entropy

Shannon entropy for PWG is given as

$$\eta_x = log(1+\lambda\sigma) - log(\lambda\alpha\sigma^2) - (\alpha-1)E(log\, x) - E\left(log\left(\lambda + e^{\lambda x^{\alpha}} - 1\right)\right)$$

$$-\lambda E(x^{\alpha}) + \sigma E\left(e^{\lambda x^{\alpha}}\right) - \sigma, \qquad (12)$$

where $E\left(e^{\lambda x^{\alpha}}\right) = \frac{\lambda\alpha\sigma^2 e^{\sigma}}{1+\lambda\sigma}\sum_{k=0}^{\infty}\frac{(-1)^k\sigma^k}{\Gamma(k+1)}\left(-\frac{(\lambda-1)}{\alpha\lambda(k+2)} - \frac{1}{\alpha\lambda(k+3)}\right)$,

$E(x^{\alpha}) = \frac{\lambda\alpha\sigma^2 e^{\sigma}}{1+\lambda\sigma}\sum_{k=0}^{\infty}\frac{(-1)^k\sigma^k}{\Gamma(k+1)}\left(\frac{(\lambda-1)}{\alpha\lambda^2(k+1)^2} + \frac{1}{\alpha\lambda^2(k+2)^2}\right)$,

and $E\left(log\left(\lambda + e^{\lambda x^{\alpha}} - 1\right)\right) = -\left(\frac{\sigma^2 e^{\sigma\lambda}}{1+\lambda\sigma}\right)\left(\frac{\xi + Log(\sigma) - 1}{\sigma^2}\right)$,

where ξ is the Euler constant and it is equal 0.577216.

The following integral

$$E(log\, x) = \int_0^{\infty} log\, x\ \frac{\lambda\alpha\sigma^2}{1+\lambda\sigma} x^{\alpha-1}\left(\lambda + e^{\lambda x^{\alpha}} - 1\right)e^{\lambda x^{\alpha} - \sigma\left(e^{\lambda x^{\alpha}} - 1\right)}\, dx,$$

cannot be given explicit solution, so Shannon entropy solved numerically.

Some numerical values for Shannon entropy are displayed in Table 2. It can be observed that Shannon entropy decreases with increasing λ and α and it is increases with increasing σ and it can have negative values.

Table 2: Shannon entropy for several arbitrary parameter values.

Parameters	σ = 0.1, α = 2	λ = 0.5, α = 2		λ = 0.5, σ = 0.1	
λ	S. Entropy	σ	S. Entropy	α	S. Entropy
1	-0.0399856	0.1	0.26914	1.8	0.466937
1.5	-0.210973	0.2	0.360073	2	0.26914
2	-0.327464	0.3	0.408451	2.8	-0.305027
2.5	-0.415203	0.4	0.43715	3	-0.413635

2.3.2. Rényi entropy

The Rényi entropy for PWG is given by

$$I_R(\gamma) = \sum_{k,i=0}^{\infty} \sum_{j=0}^{\gamma} \omega_{k,i,j} \frac{\Gamma\left(\gamma - \frac{\gamma}{\alpha} + \frac{1}{\alpha} + i\right)}{\Gamma(k+1)\Gamma(i+1)}, \tag{13}$$

where $\omega_{k,i,j} = \frac{1}{1-\gamma} log \left[\frac{\lambda\alpha\sigma^2}{1+\lambda\sigma}\right]^{\gamma} \binom{\gamma}{j} \frac{(-1)^{k+\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}+i} (\lambda-1)^{\gamma-j} (\gamma\sigma)^k e^{\sigma\gamma} \left((\gamma+k)\lambda\right)^i}{\alpha(\lambda j)^{\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}+i}}$.

Some numerical values for Rényi entropy are given in Table 3. It can be noted that Rényi entropy decreases with increasing λ and α and it increases with increasing σ.

Table 3: Rényi entropy for several arbitrary parameter values.

Parameters	σ = 0.1, α = 1.5, γ =2	λ = 0.5, α = 1.5, γ =2		λ = 0.5, σ = 0.1, γ =2	
λ	R. Entropy	σ	R. Entropy	α	R. Entropy
1	0.219417	0.1	0.652731	1	1.63878
1.5	-0.0250912	0.2	0.704905	1.5	0.652731
2	-0.193677	0.3	0.729438	2	0.0732073
2.5	-0.321513	0.4	0.740581	2.5	-0.325513

2.3.3. Mathai-Haubold entropy

The Mathai-Haubold for PWG is given by

$$J_{MH}(\delta) = \sum_{k,i=0}^{\infty} \sum_{j=0}^{2-\delta} \nu_{k,i,j} \frac{\Gamma\left(\frac{\delta-1}{\alpha} - \delta + i + 2\right)}{\Gamma(k+1)\Gamma(i+1)}, \tag{14}$$

where $\nu_{k,i,j} = \frac{1}{\delta-1} \binom{2-\delta}{j} (-1)^{k+\frac{\delta-1}{\alpha}-\delta+i+2} \frac{(\lambda-1)^{2-\delta-j} \left((2-\delta)\sigma\right)^k e^{\sigma(2-\delta)} \left((2-\delta+k)\lambda\right)^i}{\alpha(\lambda j)^{\frac{\delta-1}{\alpha}-\delta+i+2}}$.

Some numerical values for the Mathai-Haubold entropy are summarized in Table 4. It is seen that Mathai-Haubold entropy decreases with increasing λ, α and σ.

Table 4: Mathai-Haubold entropy for several arbitrary parameter values.

Parameters	σ = 0.1, α = 1.5, δ =1.5	λ = 0.5, α = 1.5, δ =1.5		λ = 0.5, σ = 0.1, δ =1.5	
λ	M-H. Entropy	σ	M-H. Entropy	α	M-H. Entropy
1	3.88548	0.1	8.75748	1.5	8.75748
1.5	2.25847	0.2	7.20391	1.7	4.1509
2	1.44356	0.3	6.24107	1.9	2.16765
2.5	0.953877	0.4	5.54189	2.1	1.18196

In the next subsections we will use the following lemma:

Lemma 1: Let

$$J(z; r, \lambda, \sigma, \alpha) = \int_0^z x^r f(x) dx$$

$$= \frac{\lambda\alpha\sigma^2 e^{\sigma}}{1+\lambda\sigma} \sum_{j=0}^{\infty} \frac{(-1)^j \sigma^j}{\Gamma(j+1)} \int_0^z x^{r+\alpha-1} \left(\lambda - 1 + e^{\lambda x^{\alpha}}\right) e^{\lambda(j+1)x^{\alpha}} dx$$

$$= \frac{\lambda\alpha\sigma^2 e^{\sigma}}{1+\lambda\sigma}\sum_{j=0}^{\infty}\frac{(-1)^j\sigma^j}{\Gamma(j+1)}\left[\int_0^z(\lambda-1)\ x^{r+\alpha-1}e^{\lambda(j+1)x^{\alpha}}dx\right.$$

$$\left.+\int_0^z x^{r+\alpha-1}\,e^{\lambda(j+2)x^{\alpha}}dx\right]$$

$$=\frac{\lambda\sigma^2e^{\sigma}}{1+\lambda\sigma}\sum_{j=0}^{\infty}\frac{(-1)^{j-\frac{r}{\alpha}-1}\sigma^j}{\lambda^{\frac{r}{\alpha}}\Gamma(j+1)}\left(\frac{1}{(j+1)^{\frac{r}{\alpha}+1}}\left(\Gamma\left(\frac{r}{\alpha}+1\right)\right.\right.$$

$$\left.-\Gamma\left(\frac{r}{\alpha}+1,-(j+1)z^{\alpha}\lambda\right)\right.$$

$$\left.+\frac{1}{(j+2)^{\frac{r}{\alpha}+1}}\left(\Gamma\left(\frac{r}{\alpha}+1\right)-\Gamma\left(\frac{r}{\alpha}+1,-(j+2)z^{\alpha}\lambda\right)\right)\right). \qquad (15)$$

2.4. Residual life and reversed residual life functions

Residual life and reversed residual life random variables are used extensively in reliability analysis and the risk theory. Here, we investigate some of their related statistical functions, such as the survival function, mean and variance in connection with the PWG distribution.

2.4.1. Residual lifetime function

The residual life is the period from time t until the time of failure and it is defined by the conditional random variable $R_{(t)} \coloneqq X-t \mid X>t, t\geq 0$.

The survival function of the residual lifetime $R_{(t)}$ for the PWG distribution is:

$$S_{R_{(t)}}(x)=\frac{\left(1+\lambda\sigma+\sigma\left(e^{\lambda(x+t)^{\alpha}}-1\right)\right)}{\left(1+\lambda\sigma+\sigma\left(e^{\lambda t^{\alpha}}-1\right)\right)}e^{-\sigma e^{\lambda t^{\alpha}}\left(e^{\lambda x^{\alpha}}-1\right)}. \qquad (16)$$

The pdf and the hazard rate function of $R_{(t)}$ are respectively, given as:

$$f_{R_{(t)}}(x)=\frac{\lambda\alpha\sigma^2(x+t)^{\alpha-1}\left(\lambda+e^{\lambda(x+t)^{\alpha}}-1\right)}{\left(1+\lambda\sigma+\sigma\left(e^{\lambda t^{\alpha}}-1\right)\right)}e^{\lambda(x+t)^{\alpha}-\sigma e^{\lambda t^{\alpha}}\left(e^{\lambda x^{\alpha}}-1\right)}, \qquad (17)$$

$$h_{R_{(t)}}(x)=\frac{\lambda\alpha\sigma^2(x+t)^{\alpha-1}\left(\lambda+e^{\lambda(x+t)^{\alpha}}-1\right)}{1+\sigma\left(\lambda+e^{\lambda(x+t)^{\alpha}}-1\right)}e^{\lambda(x+t)^{\alpha}}. \qquad (18)$$

The mean of $R_{(t)}$ for the PWG distribution is:

$$K(t)=\frac{1}{S(t)}\left(E(X)-J(t;1,\lambda,\sigma,\alpha)\right)-t,\ \ t\geq 0, \qquad (19)$$

where *E(x)* can be obtained by using equation (8), *S(t)* can be obtained from equation (4) and $J(t;1,\lambda,\sigma,\alpha)$ is computed by lemma 2.1 for *r=1*.

The variance of $R_{(t)}$ for the PWG distribution is:

$$V(t)=\frac{1}{S(t)}\left(E(X^2)-J(t;2,\lambda,\sigma,\alpha)\right)-t^2-2tK(t)-\left(K(t)\right)^2, \qquad (20)$$

where $E(x^2)$ can be obtain by using equation (9), *S(t)* can be obtained from equation

(4) and $J(t;2,\lambda,\sigma,\alpha)$ is computed by lemma 2.1 for *r=2*.

Some numerical values of the mean residual life are displayed in Table 5 for a set of arbitrary choices of the parameters λ, α and σ at the time points t = 2, 4, 6. This table shows that the mean residual life decreases with increasing the time points t, and decreases with increasing λ, α and σ.

2.4.2. Reversed residual life function

The reversed residual life is the time elapsed from the failure of a component given that its life satisfies $X\leq t$, and it is defined as the conditional random variable $\bar{R}(t)\coloneqq t-X\mid X\leq t$.

The survival function of the $\bar{R}(t)$ for the PWG distribution is:

$$S_{\bar{R}(t)}(x)=\frac{1+\lambda\sigma-\left(1+\lambda\sigma+\sigma\left(e^{\lambda(t-x)^{\alpha}}-1\right)\right)e^{-\sigma\left(e^{\lambda(t-x)^{\alpha}}-1\right)}}{1+\lambda\sigma-\left(1+\lambda\sigma+\sigma\left(e^{\lambda t^{\alpha}}-1\right)\right)e^{-\sigma\left(e^{\lambda t^{\alpha}}-1\right)}}. \qquad (21)$$

The pdf and the hazard rate function of $\bar{R}(t)$ are:

$$f_{\bar{R}(t)}(x) = \frac{\lambda\alpha\sigma^2(t-x)^{\alpha-1}\left(\lambda+e^{\lambda(t-x)^\alpha}-1\right)e^{\lambda(t-x)^\alpha-\sigma\left(e^{\lambda(t-x)^\alpha}-1\right)}}{1+\lambda\sigma-\left(1+\lambda\sigma+\sigma\left(e^{\lambda t^\alpha}-1\right)\right)e^{-\sigma\left(e^{\lambda t^\alpha}-1\right)}}, \quad (22)$$

$$h_{\bar{R}(t)}(x) = \frac{\lambda\alpha\sigma^2(t-x)^{\alpha-1}\left(\lambda+e^{\lambda(t-x)^\alpha}-1\right)e^{\lambda(t-x)^\alpha-\sigma\left(e^{\lambda(t-x)^\alpha}-1\right)}}{1+\lambda\sigma-\left(1+\lambda\sigma+\sigma\left(e^{\lambda(t-x)^\alpha}-1\right)\right)e^{-\sigma\left(e^{\lambda(t-x)^\alpha}-1\right)}}. \quad (23)$$

The mean of $\bar{R}(t)$ for the PWG distribution is given by:

$$L(t) = t - \frac{J(t;1,\lambda,\sigma,\alpha)}{F(t)}, \quad (24)$$

where *F(t)* can be obtained from equation (4) and $J(t;1,\lambda,\sigma,\alpha)$ is computed by lemma 1 for *r=1*.

The variance of $\bar{R}(t)$ for the PWG distribution is given by:

$$W(t) = 2tL(t) - \left(L(t)\right)^2 - t^2 + \frac{J(t;2,\lambda,\sigma,\alpha)}{F(t)}, \quad (25)$$

where $J(t;2,\lambda,\sigma,\alpha)$ is computed by lemma 1 for *r=2*.

In Table 6 we give some numerical values for the mean reversed life with arbitrary choices of the parameters λ, α and σ at the time points t = 2, 4, 6. It can be seen that the mean reversed residual life increases with increasing the time points t and increases with increasing σ and decreases with increasing λ and α.

Table 5: Mean residual life function for several arbitrary parameter values.

Parameters	λ= 0.5 , σ = 0.5		
A	t = 2	t = 4	t = 6
0.5	8.19547	7.49908	6.87677
0.7	3.23152	2.3491	1.71631
0.9	1.69539	0.866969	0.429601
1.1	1.00168	0.310067	0.0837276
Parameters	λ= 0.8 , α = 1.2		
Σ	t = 2	t = 4	t = 6
0.1	0.959062	0.113509	0.00756708
0.3	0.422983	0.0384394	0.00252301
0.5	0.26988	0.0231087	0.00151387
0.7	0.197582	0.0165173	0.00108135
Parameters	α = 0.5 , σ = 0.1		
Λ	t = 2	t = 4	t = 6
0.8	11.1475	9.67559	8.38155
0.9	8.46353	7.07045	5.89531
1	6.5589	5.25014	4.19798
1.1	5.1635	3.94346	3.01468

Table 6: Mean reversed residual life function for several arbitrary parameter values.

Parameters	λ= 0.5 , σ = 0.5		
A	t = 2	t = 4	t = 6
0.1	1.73538	3.46199	5.18519
0.3	1.36932	2.68994	3.99456
0.5	1.12278	2.16213	3.20233
0.7	0.94384	1.8055	2.80479
Parameters	λ= 0.8 , α = 1.2		
σ	t = 2	t = 4	t = 6
0.1	0.473489	1.22738	3.2157
0.3	0.594419	2.11939	4.11939
0.5	0.725105	2.52047	4.52047
0.7	0.854786	2.76694	4.76694
Parameters	α = 0.5 , σ = 0.1		
λ	t = 2	t = 4	t = 6

0.1	1.16637	2.2501	3.29594
0.3	1.10898	2.08723	3.00025
0.5	1.05302	1.93473	2.73544
0.7	0.999689	1.79996	2.52209

2.5. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves have many applications in economics to study income and poverty, reliability, medicine and insurance. The list of applications in other areas: diseases risk to optimize health benefits under constrains, seasonal variation of environmental radon gas and statistical nonuniformity of sediment transport rate.

The Bonferroni curve $B[F(x)]$ for the PWG distribution is defined by:

$$B[F(x)] == \frac{1}{E(x)F(x)} J(x;1,\lambda,\sigma,\alpha) \qquad (26)$$

where *E(x)* can be obtained by using equation (8), *F(x)* can be obtained from equation

(4), and $J(x;1,\lambda,\sigma,\alpha)$ is computed by lemma 1 for *r=1*.

Also, the Lorenz curve of F(.) that follows of the PWG distribution is the graph of

$$L[F(x)] = \frac{J(x;1,\lambda,\sigma,\alpha)}{\sum_{k=0}^{\infty} \frac{(-1)^{\frac{1}{\alpha}+k+1} \sigma^{k+2} e^{\sigma}}{(1+\lambda\sigma)\lambda^{\frac{1}{\alpha}} \Gamma(k+1)} \Gamma\left(\frac{1}{\alpha}+1\right) \left[\frac{\lambda-1}{(k+1)^{\frac{1}{\alpha}+1}} + \frac{1}{(k+2)^{\frac{1}{\alpha}+1}}\right]}, \qquad (27)$$

where $J(x;1,\lambda,\sigma,\alpha)$ is computed by lemma 1 for *r=1*.

3. Statistical inferences

3.1. Maximum likelihood estimates

In this sub-section, the method of maximum likelihood is studied to estimate the unknown parameters of the PWG model.

Let $x_1, x_2, \ldots, x_n$be a random sample of size $n$from the PWG distribution with parameters λ, α and σ. Then, the corresponding log-likelihood function is

$$\ell = n\,log(\lambda) + n\,log(\alpha) + 2n\,log(\sigma) - n\,log(1+\lambda\sigma) + (\alpha-1)\sum_{i=1}^{n} log\,x_i$$

$$+\lambda \sum_{i=1}^{n} x_i^{\alpha} - \sigma \sum_{i=1}^{n} \left(e^{\lambda x_i^{\alpha}} - 1\right) + \sum_{i=1}^{n} log\left(\lambda + e^{\lambda x_i^{\alpha}} - 1\right). \qquad (28)$$

Differentiating Eq. (28) with respect to λ, α and σ, we obtain the following equations:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \frac{n\sigma}{1+\lambda\sigma} + \sum_{i=1}^{n} x_i^{\alpha} - \sigma \sum_{i=1}^{n} x_i^{\alpha} e^{\lambda x_i^{\alpha}} + \sum_{i=1}^{n} \frac{x_i^{\alpha} e^{\lambda x_i^{\alpha}} + 1}{\left(\lambda + e^{\lambda x_i^{\alpha}} - 1\right)} = 0, \qquad (29)$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\,x_i + \lambda \sum_{i=1}^{n} x_i^{\alpha} \,log\,x_i - \sigma\lambda \sum_{i=1}^{n} x_i^{\alpha} e^{\lambda x_i^{\alpha}} \,log\,x_i$$

$$+\lambda \sum_{i=1}^{n} \frac{x_i^{\alpha} e^{\lambda x_i^{\alpha}} \,log\,x_i}{\left(\lambda + e^{\lambda x_i^{\alpha}} - 1\right)} = 0 \qquad (30)$$

and

$$\frac{\partial \ell}{\partial \sigma} = \frac{2n}{\sigma} - \frac{n\lambda}{1+\lambda\sigma} - \sum_{i=1}^{n} \left(e^{\lambda x_i^{\alpha}} - 1\right) = 0. \qquad (31)$$

The maximum likelihood estimators (MLEs) $\hat{\lambda}, \hat{\alpha}$ and $\hat{\sigma}$ of λ, α and σ, respectively, can be obtained by solving the above nonlinear equations numerically using the statistical software Mathematic package.

3.2. Fisher's information matrix

In order to determine the confidence intervals for the distribution parameters, we need to construct the information matrix. The corresponding 3×3 observed information matrix $I_n(\alpha, \lambda, \sigma)$ is

$$I_n = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} & \frac{\partial^2 \ell}{\partial \alpha \partial \sigma} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \sigma} \\ \frac{\partial^2 \ell}{\partial \sigma \partial \alpha} & \frac{\partial^2 \ell}{\partial \sigma \partial \lambda} & \frac{\partial^2 \ell}{\partial \sigma^2} \end{pmatrix} = \begin{pmatrix} I_{11} I_{12} I_{13} \\ I_{21} I_{22} I_{23} \\ I_{31} I_{32} I_{33} \end{pmatrix}.$$

The elements of $I_n$are given by

$$I_{11} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n} x_i^\alpha (\log x_i)^2 - \sigma\lambda \sum_{i=1}^{n} e^{\lambda x_i^\alpha} x_i^\alpha (1 + \lambda x_i^\alpha)(\log x_i)^2$$

$$+\lambda \sum_{i=1}^{n} \left(\frac{e^{\lambda x_i^\alpha} x_i^\alpha (1 + \lambda x_i^\alpha)(\log x_i)^2}{\lambda + e^{\lambda x_i^\alpha} - 1} - \lambda \frac{e^{2\lambda x_i^\alpha} x_i^{2\alpha} (\log x_i)^2}{\left(\lambda + e^{\lambda x_i^\alpha} - 1\right)^2} \right),$$

$$I_{12} = \sum_{i=1}^{n} x_i^\alpha \log x_i - \sigma \sum_{i=1}^{n} e^{\lambda x_i^\alpha} x_i^\alpha (1 + \lambda x_i^\alpha) \log x_i$$

$$+ \sum_{i=1}^{n} \left(\frac{x_i^\alpha e^{\lambda x_i^\alpha} (1 + \lambda x_i^\alpha) \log x_i}{\lambda + e^{\lambda x_i^\alpha} - 1} - \lambda \frac{e^{\lambda x_i^\alpha} \left(1 + x_i^\alpha e^{\lambda x_i^\alpha}\right) x_i^\alpha \log x_i}{\left(\lambda + e^{\lambda x_i^\alpha} - 1\right)^2} \right),$$

$$I_{13} = -\lambda \sum_{i=1}^{n} x_i^\alpha e^{\lambda x_i^\alpha} \log x_i,$$

$$I_{22} = -\frac{n}{\lambda^2} + \frac{n\sigma^2}{(1+\lambda\sigma)^2} - \sigma \sum_{i=1}^{n} x_i^{2\alpha} e^{\lambda x_i^\alpha}$$

$$+ \sum_{i=1}^{n} \left(\frac{x_i^{2\alpha} e^{\lambda x_i^\alpha}}{\lambda + e^{\lambda x_i^\alpha} - 1} - \frac{x_i^\alpha e^{\lambda x_i^\alpha} \left(1 + x_i^\alpha e^{\lambda x_i^\alpha}\right)}{\left(\lambda + e^{\lambda x_i^\alpha} - 1\right)^2} \right),$$

$$I_{23} = \frac{n\lambda\sigma}{(1+\lambda\sigma)^2} - \frac{n}{1+\lambda\sigma} - \sum_{i=1}^{n} x_i^\alpha e^{\lambda x_i^\alpha},$$

and

$$I_{33} = \frac{n\lambda^2}{(1+\lambda\sigma)^2} - \frac{2n}{\sigma^2}.$$

3.3. Approximate confidence intervals

Since the MLEs of α, λ and σ cannot be determined in closed forms, it is not easy to obtain the exact confidence intervals for α, λ and σ. Hence, we can use the asymptotic behavior of the MLE to obtain the asymptotic confidence intervals for the model parameters. Using large sample approximation, the asymptotic distribution of the MLE is $\left[\sqrt{n}(\hat{\alpha} - \alpha), \sqrt{n}\left(\hat{\lambda} - \lambda\right), \sqrt{n}(\hat{\sigma} - \sigma)\right] \to N_3(0, \Delta^{-1})$, where $\Delta^{-1}$is the inverse of observed information matrix. The estimated variance covariance matrix of the parameters α, λ and σ can be obtained as follows

$$\Delta^{-1} = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} - \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} - \frac{\partial^2 \ell}{\partial \alpha \partial \sigma} \\ -\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} - \frac{\partial^2 \ell}{\partial \lambda^2} - \frac{\partial^2 \ell}{\partial \lambda \partial \sigma} \\ -\frac{\partial^2 \ell}{\partial \sigma \partial \alpha} - \frac{\partial^2 \ell}{\partial \sigma \partial \lambda} - \frac{\partial^2 \ell}{\partial \sigma^2} \end{pmatrix}^{-1} = \begin{pmatrix} \Delta_{11} \Delta_{12} \Delta_{13} \\ \Delta_{21} \Delta_{22} \Delta_{23} \\ \Delta_{31} \Delta_{32} \Delta_{33} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} var(\hat{\alpha})\ cov\left(\hat{\alpha}, \hat{\lambda}\right)\ cov(\hat{\alpha}, \hat{\sigma}) \\ cov\left(\hat{\lambda}, \hat{\alpha}\right)\ var\left(\hat{\lambda}\right)\ cov\left(\hat{\lambda}, \hat{\sigma}\right) \\ cov(\hat{\sigma}, \hat{\alpha})\ cov\left(\hat{\sigma}, \hat{\lambda}\right)\ var(\hat{\sigma}) \end{pmatrix}.$$

Hence, the asymptotic $100(1-\delta)\%$ confidence intervals for α, λ and σ are given by

$$\left\{ \begin{matrix} \hat{\alpha}_L = \hat{\alpha} - Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\alpha})}, \hat{\alpha}_U = \hat{\alpha} + Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\alpha})} \\ \hat{\lambda}_L = \hat{\lambda} - Z_{\frac{\delta}{2}} \sqrt{Var\left(\hat{\lambda}\right)}, \hat{\lambda}_U = \hat{\lambda} + Z_{\frac{\delta}{2}} \sqrt{Var\left(\hat{\lambda}\right)} \\ \hat{\sigma}_L = \hat{\sigma} - Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\sigma})}, \hat{\sigma}_U = \hat{\sigma} + Z_{\frac{\delta}{2}} \sqrt{Var(\hat{\sigma})} \end{matrix} \right\},$$

where $Z_{\frac{\delta}{2}}$ is the upper $100(1-\delta)\%$ percentile of the standard normal distribution.

4. Data analysis

By making use of two real data set, we illustrate the applicability of the PWG distribution among a set of classical and recent distributions, based on a set of goodness-of-fit statistics. We estimate the model parameters by using the maximum likelihood method. We compare goodness-of-fit of the models with the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC) goodness -of-fit statistics. Further, we get the Kolmogorov-Smirnov (K-S) statistic with its corresponding p- value. In general, the model has the smaller values of these statistics and the largest value of the p-value is the best model to fit the data.

4.1. Strength of 1.5 cm glass fibers data

The data set is given in Table 7. This data set consists of 63 observations of the strength of 1.5 cm glass fibers measures at the UK National Physical Laboratory. These data have been analyzed by Smith and Naylor [13]. Table 8 gives some descriptive statistics for the data set and using it we note that the data are under-dispersed (variance≺mean) and positively skewed. Further, the data set having positive kurtosis, that is the tail of their histograms increases quickly, and hence its histogram tail increases slowly. Description of such data go with the features of the PWG distribution and this proves the suitability of this distribution for analyzing such data. This suitability is confirmed again by noting the closeness of those descriptive statistics with their theoretical measures given by Table 9 of the PWG distribution for the strength 1.5 glass fibers data. On the other side, comparing the PWG distribution with other classical and recent distributions is done as follows.

The pdfs of the compared models are given by:

- The 3- parameter Gompertz (3-PG) distribution (Haile et al. [7]) with density function

$$f(x)=\left(\beta e^{\alpha x+\eta e^{\alpha x}}\right) exp\left\{-\frac{\beta}{\alpha\eta}\left(e^{\eta\alpha x}-e^{\eta}\right)\right\}, x \succ 0, \beta \succ 0, \alpha,\eta \text{ both real and finite.}$$

- The generalized Gompertz (GG) distribution (El-Gohary et al. [4]) with density function

$$f(x)=\theta\lambda e^{cx}e^{-\frac{\lambda}{c}(e^{cx}-1)}\left[1-e^{-\frac{\lambda}{c}(e^{cx}-1)}\right]^{\theta-1}, x\geq 0, \lambda,\theta>0, c\geq 0.$$

- The transmuted Gompertz (TG) distribution (Khan et al. [9]) with density function

$$f(x)=\alpha e^{\beta x} exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}\left[1-\lambda+2\lambda\, exp\left\{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)\right\}\right], x \succ 0, \alpha,\beta \succ 0, |\lambda|\leq 1.$$

- The power Lomax (PLo) distribution (Rady et al. [11]) with density function

$$f(x)=\alpha\beta\lambda^{\alpha}x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-\alpha-1}, x \succ 0, \alpha,\beta,\lambda>0.$$

- The beta – Gompertz (BG) distribution (Jafari et. al. [8]) with density function

$$f(x)=\frac{1}{B(\alpha,\beta)}\theta e^{\gamma x} exp\left\{-\frac{\beta\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}\left[1-exp\left\{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}\right]^{\alpha-1},$$

$x\geq 0, \alpha,\beta,\theta,\gamma>0.$

For the data set, we estimate the unknown parameters of each distribution by the maximum-likelihood method, and using those estimates, we obtain the statistics K-S, p-value, AIC, BIC and HQIC. The obtained results are reported in Tables 10 and 11. From these tables, the smallest values of the K-S, AIC, BIC and HQIC and the largest value of p-value are obtained for the PWG distribution. Hence, we conclude that the PWG distribution provides the best fit among the compared distributions. This result is asserted graphically by Figure 2, where the estimated densities and estimated survival functions for the compared distributions of the data set are plotted based on the density and survival functions of each distribution and replacing the parameters with their MLEs given in Table 10. To show that the likelihood equations have a unique solution for the parameters of the PWG distribution, we plot the profiles of the log-likelihood functions of α,σ and λ for the strength of 1.5 cm glass fibers data in Figures 3. Those figures confirm this fact.

Table 7: Strength of 1.5 cm glass fibers data.

0.55	0.74	0.77	0.81	0.84	0.93	1.04	1.11	1.13	1.24	1.25	1.27	1.28	1.29	1.30	1.36	1.39	1.42	1.48	1.48	1.49
1.50	1.50	1.51	1.52	1.53	1.54	1.55	1.55	1.58	1.59	1.60	1.61	1.61	1.61	1.61	1.62	1.62	1.63	1.64	1.66	
1.66	1.66	1.67	1.68	1.68	1.69	2.00	2.01	2.24												

Table 8: Descriptive statistics of the data set.

Mean	Median	SD	SK	KS	MD-mean	MD-median	SE

1.4408	1.525	0.33012	0.622915	0.64624	0.24857	0.2328	1.56197

MD:= Mean deviation, KS:= kurtosis, SK:= skewness, SE:= Shannon entropy

Table 9: Some measures of the PWG distribution for the strength of 1.5 cm glass fibers data.

Mean	Median	SD	MD-mean	MD-median	SE	RE
1.4437	1.46427	0.3158	0.47374	0.4437	0.25486	0.09678

SE:= Shannon entropy, RE:= Rényi entropy

Table 10: The MLEs of the parameters of some models fitted to the strength of 1.5 cm glass fibers data.

Distributions	Estimates			
$3-PG(\beta,\alpha,\eta)$	0.007368	0.548185	2.1219	-
$GG(\lambda,\theta,c)$	0.11975	2.28075	0.11975	-
$TG(\alpha,\beta,\lambda)$	0.006674	3.63212	0.805819	-
$PLo(\alpha,\beta,\lambda)$	11.7034	5.5327	130.762	-
$BG(\alpha,\beta,\theta,\gamma)$	2.31688	1.12869	0.11473	2.17726
$PWG(\alpha,\lambda,\sigma)$	4.09893	0.002535	118.114	-

Table 11: The values of K-S, p- value, AIC, BIC and HQIC statistics for some models fitted to the strength of 1.5 cm glass fibers data.

Distribution	K-S value	p-value	AIC	BIC	HQIC
$3-PG(\beta,\alpha,\eta)$	0.21441	0.02015	42.9613	48.6974	45.1456
$GG(\lambda,\theta,c)$	0.16919	0.114216	33.368	39.1048	35.553
$TG(\alpha,\beta,\lambda)$	0.17040	0.10959	33.516	39.2522	35.7005
$PLo(\alpha,\beta,\lambda)$	0.17142	0.10587	33.686	39.4223	35.8706
$BG(\alpha,\beta,\theta,\gamma)$	0.16890	0.11537	35.363	43.0117	38.276
$PWG(\alpha,\lambda,\sigma)$	**0.16266**	**0.14182**	**32.9907**	**38.7268**	**35.185**

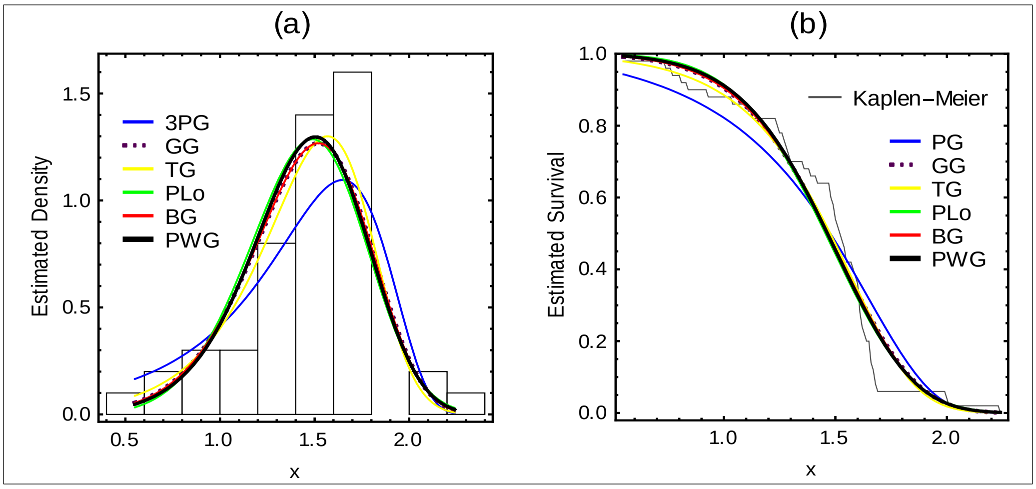

Figure 2: Estimated densities and survival functions for the considered distributions for the strength of 1.5 cm glass of fibers.

Consequently, the variance-covariance matrix of the MLEs of the PWG distribution for the strength of 1.5 cm glass fibers data is:

$$\begin{pmatrix} 0.18672 & -0.00225 & 96.0427 \\ -0.00225 & 0.0001313 & -6.0647 \\ 96.0427 & -6.06473 & 280525.0 \end{pmatrix}.$$

We note that the diagonal entries of above matrix are the variances of the MLEs of the PWG parameters $\alpha, \lambda$and $\sigma$of the data set while the values -0.002255, 96.0427, and -6.06473 represent the covariance between the MLEs of ($\alpha$and λ), ($\alpha$and σ), and ($\lambda$and σ), for the data set, respectively.

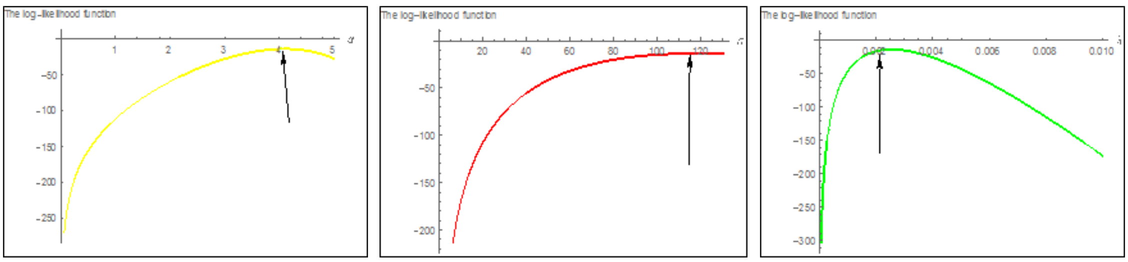

Figure 3: The profile of the log-likelihood as a function of α, σ and λ for the PWG model fitted to the strength of 1.5 cm glass fibers data.

4.2. Growth hormone data

The data set is given in Table 12. This data set consists of 35 observations of the Growth hormone data set. Children of the Program Hormonal (de Crescimento da Secretaria da Sa´ude de Minas Gerais) were diagnosed with growth hormone deficiency. The data consists of the estimated time since the growth hormone medication until the children reached the target height. The data set have been analyzed by De Morais [3]. Table 13 shows some descriptive statistics for Growth hormone data and it is noted that the data set have positive kurtosis and skewness. In Table 14 there are some of the corresponding theoretical measures of the PWG distribution of the data set. From Tables 13 and 14 it can be concluded that the considered measures of the PWG distribution are close to the sample measures given by Table 13 for the data set.

Comparing the PWG distribution with other classical and recent distributions is done as follows:

The pdfs of the compared models are given by:

- The Gompertz (G) distribution (Gompertz [6]) with density function

$$f(x) = \lambda\sigma e^{\lambda x-\sigma(e^{\lambda x}-1)}, x > 0\ , 0 < \lambda < 1\ , \sigma > 0.$$

- The Shifted Gompertz (SG) distribution (Bemmaor [2]) with density function

$$f(x) = \beta e^{-(\beta x+\alpha e^{-\beta x})}\left(1+\alpha\left(1-e^{-\beta x}\right)\right), x > 0, \alpha > 0, \beta > 0.$$

- The Gompertz Lomax (GL) distribution (Pelumi et al. [10]) with density function

$$f(x) = \theta\alpha\beta(1+\beta x)^{\alpha\gamma-1} Exp\left[\frac{\theta}{\gamma}(1-(1+\beta x)^{\alpha\gamma})\right], x > 0, \theta, \gamma, \alpha, \beta > 0.$$

For the data set, we estimate the unknown parameters of each distribution by the maximum-likelihood method, and by using those estimates, we obtain the statistics K-S, p-value, AIC, BIC and HQIC. From Tables 15 and 16, the smallest values of the K-S, AIC, BIC and HQIC and the largest value of p-value are obtained for the PWG distribution. So, we conclude that the PWG distribution provides the best fit among the compared distributions. Figure 6 confirm this result where the estimated densities and estimated survival functions for the compared distributions of the data set are plotted based on the density and survival functions of each distribution and replacing the parameters with their MLEs given in Table 15. We plot the profiles of the log-likelihood functions of α, σ and λ for the growth hormone data in Figure 7, this figure confirm that the likelihood equations have a unique solution for the parameters of the PWG distribution.

Table 12: Growth hormone data.

2.15	2.20	2.55	2.56	2.63	2.74	2.81	2.90	3.05	3.41	3.43	3.43	3.84	4.16	4.18	4.36	4.42	4.51	4.60	4.61	4.75	5.03	5.10
5.44	5.90	5.96	6.77	7.82	8.00	8.16	8.21	8.72	10.40	13.20	13.70											

Table 13: Descriptive statistics of the data set.

Mean	**Median**	**SD**	**SK**	**KS**	**MD-mean**	**MD-median**	**SE**
5.30571	4.51	2.91125	1.3509	1.27507	2.20637	2.04429	1.52687

MD:= Mean deviation, KS:= kurtosis, SK:= skewness, SE:= Shannon entropy

Table 14: Some measures of the PWG distribution for the growth hormone data.

Mean	**Median**	**SD**	**MD-mean**	**MD-median**	**SE**	**RE**
5.27519	4.9406	2.91226	0.543195	4.27519	2.43026	2.31079

SE:= Shannon entropy, RE:= Rényi entropy

Table 15: The MLEs of the parameters of some models fitted to the growth hormone data.

Distributions	**Estimates**			
G(λ, σ)	0.176491	0.501371	-	-
SG(α, β)	1.	0.282078	-	-
GL(θ, γ, α, β)	0.0142943	2.91868	0.728317	1.8505
PWG(α, λ, σ)	1.22191	0.0303906	6.33536	-

Table 16: The values of K-S, p- value, AIC, BIC and HQIC statistics for some models fitted to the growth hormone data.

Distribution	**K-S value**	**p-value**	**AIC**	**BIC**	**HQIC**
G(λ, σ)	0.20656	0.100895	178.194	181.304	179.267
SG(α, β)	0.263597	0.0154415	179.551	182.662	180.625
GL(θ, γ, α, β)	0.147985	0.427445	174.559	180.78	176.706
PWG(α, λ, σ)	**0.142727**	**0.473893**	**172.496**	**177.162**	**174.107**

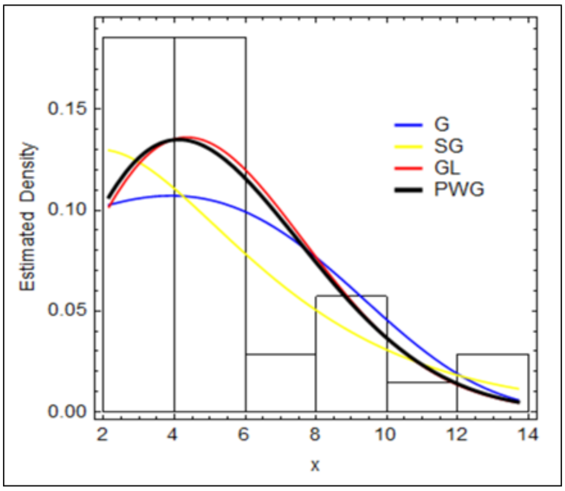

Figure 6: Estimated densities functions for the considered distributions for the growth hormone data.

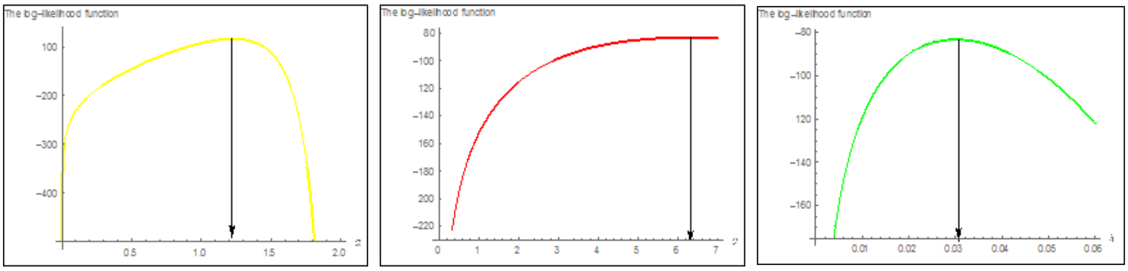

Figure 7: The profile of the log-likelihood as a function of α ,σ and λ for the PWG model fitted to the growth hormone data.

5. Concluding remarks

We propose a new- three parameter distribution referred to as the power weighted Gompertz (PWG) distribution, which is based on the weighted Gompertz distribution. Various properties of the PWG have been derived, including the moments, moment generating function, three popular entropies, some measures of residual lifetime and reversed residual lifetime and Bonferroni and Lorenz curves. The model parameters are estimated by the method of maximum likelihood and the observed information matrix is derived. Finally, an application of the PWG distribution to two real data sets is provided to illustrate that this distribution provides a better fit than some other very well-known distributions.

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